

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
SOLUTION SKETCHES TO HOMEWORK 0

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Homework 0.1 (Warmup). Give an example of the following objects or show they cannot exist, respectively.

- a. A holomorphic function $f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ such that $f'(z) = 1/z$.
- b. A holomorphic function $f: \mathbb{C} \setminus \{\Re \leq 0\} \rightarrow \mathbb{C}$ with $f'(z) = 1/z$? Here $\{\Re \leq 0\}$ designates the set of all complex numbers with nonpositive real part.
- c. A domain $D \subset \mathbb{C}$ and a nonconstant holomorphic function $f: D \rightarrow \mathbb{C}$ such that $|f|$ is constant.
- d. A nonconstant holomorphic function $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $f(n) = 0$ for every $n \in \mathbb{N}$.
- e. A nonconstant holomorphic function $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $f(i + 1/n) = 0$ for every $n \in \mathbb{N}$.
- f. A nonconstant holomorphic function $f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ that is bounded.

Solution. a. *No.* Otherwise, by the fundamental theorem of calculus and the definition of complex path integrals we would obtain the contradiction

$$0 = f(1) - f(1) = \int_{\partial B_1(0)} f'(z) dz = \int_0^{2\pi} e^{-it} e^{it} i dt = 2\pi i.$$

b. *Yes.* On star-shaped domains every holomorphic function possesses a primitive. One function is the complex logarithm given by $f(z) := \log(z)$ (more precisely, its principal value with imaginary part in $(-\pi, \pi]$).

c. *No.* Such a function would satisfy $f(D) \subset \partial B_r(0)$ for some $r \geq 0$. In particular, the set $f(D)$ is not open, which contradicts the open mapping theorem.

d. *Yes.* Consider the function given by $f(z) := \sin(\pi z)$.

e. *No.* As the sequence $(i + 1/n)_{n \in \mathbb{N}}$ has the accumulation point i the identity theorem yields $f(z) = 0$ for every $z \in \mathbb{C}$.

f. *No.* Since the origin 0 is an isolated singularity, Riemann's lemma implies it is a removable singularity, hence f can be extended to 0 . However, the resulting extension is still bounded yet nonconstant, contradicting Liouville's theorem.

Homework 0.2 (Image of entire functions). Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a non-constant entire function (i.e. f is holomorphic on the entire complex plane). Show $f(\mathbb{C})$ is dense in $\mathbb{C}$ ¹.

Solution. Assume $f(\mathbb{C})$ is not dense in $\mathbb{C}$. Then there exists $z_0 \in \mathbb{C}$ which cannot be approximated by a sequence in $f(\mathbb{C})$; that is (check this logical step on your own), there exists $\delta > 0$ with $|f(z) - z_0| > \delta$ for every $z \in \mathbb{C}$. Then the function $g: \mathbb{C} \rightarrow \mathbb{C}$ given by

$$g(z) := \frac{1}{f(z) - z_0}$$

satisfies $|g(z)| \leq 1/\delta$. Hence by Liouville's theorem g has to be constant, which contradicts the assumption that f is nonconstant.

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¹**Hint.** Argue by contradiction and use Liouville's theorem for a suitable function.

Homework 0.3 (Complex Arzelà–Ascoli theorem). Let $K \subset \mathbf{C}$ be a compact set and let $(f_n)_{n \in \mathbf{N}}$ be a *uniformly bounded* sequence of functions $f_n: K \rightarrow \mathbf{C}$, i.e.

$$\sup_{n \in \mathbf{N}} \sup_{z \in K} |f_n(z)| < \infty.$$

- Let $S \subset K$ be a countable set. Show there exists a subsequence $(f_{n_k})_{k \in \mathbf{N}}$ such that the point evaluations $(f_{n_k}(z))_{k \in \mathbf{N}}$ converge to some value $f_z \in \mathbf{C}$ for every $z \in S$.
- Now assume in addition that $(f_n)_{n \in \mathbf{N}}$ is *equicontinuous*, i.e. for every $\varepsilon > 0$ and every $z \in K$ there exists $\delta > 0$ such that for every $x \in K$,

$$|x - z| \leq \delta \implies \sup_{n \in \mathbf{N}} |f_n(x) - f_n(z)| \leq \varepsilon.$$

Suppose moreover the set S from a. is dense in K . Show that the point evaluations $(f_{n_k}(z))_{k \in \mathbf{N}}$ converge to some value $f_z \in \mathbf{C}$ for every $z \in K$.

- Retain the setting of b. and define the function $f: K \rightarrow \mathbf{C}$ by $f(z) := f_z$. Show $(f_{n_k})_{k \in \mathbf{N}}$ converges uniformly to f on K . Conclude f is continuous.

Solution. a. Let us write $S := \{z_j : j \in \mathbf{N}\}$. Since the sequence $(f_n(z_1))_{n \in \mathbf{N}}$ is bounded, the Bolzano–Weierstraß theorem implies the existence of a subsequence $(n_{k,1})_{k \in \mathbf{N}}$ such that $(f_{n_{k,1}}(z_1))_{k \in \mathbf{N}}$ converges to some value $f_{z_1} \in \mathbf{C}$. In the second step, we note that the sequence $(f_{n_{k,1}}(z_2))_{k \in \mathbf{N}}$ is again bounded; by the same argument as before, we find another subsequence $(n_{k,2})_{k \in \mathbf{N}}$ of the previous subsequence such that $(f_{n_{k,2}}(z_2))_{k \in \mathbf{N}}$ converges to some value $f_{z_2} \in \mathbf{C}$. In the j -th step, where j is larger than two, we choose a subsequence $(n_{k,j})_{k \in \mathbf{N}}$ of all previous subsequences such that $(f_{n_{k,j}})_{k \in \mathbf{N}}$ converges to some value $f_{z_j} \in \mathbf{C}$.

It is tempting to finish the argument by induction. However, it is not clear that any sequence survives this infinite procedure. Hence we use the so-called “diagonal argument”. For $k \in \mathbf{N}$ we set $n_k := n_{k,k}$. Then the sequence $(f_{n_k}(z_j))_{k \in \mathbf{N}}$ converges to f_{z_j} for every $j \in \mathbf{N}$, since except for finitely many members, $(n_k)_{k \in \mathbf{N}}$ is a subsequence of $(n_{k,j})_{k \in \mathbf{N}}$.

b. We claim $\{f_{n_k}(z)\}_{k \in \mathbf{N}}$ constitutes a Cauchy sequence for every given $z \in K$. To reduce notation, we skip the subscript k . Given any $\varepsilon > 0$, the hypothesized equicontinuity implies the existence of $\delta_{\varepsilon,z} > 0$ such that

$$|x - z| \leq \delta_{\varepsilon,z} \implies \sup_{n \in \mathbf{N}} |f_n(x) - f_n(z)| \leq \frac{\varepsilon}{3}. \quad (0.1)$$

On the other hand, given this number $\delta_{\varepsilon,z}$, since S is dense in K we may and will choose $z^* \in S$ such that $|z - z^*| \leq \delta_{\varepsilon,z}$.

For every $m \geq n$, using (0.1) twice and the choice of z^* we observe

$$\begin{aligned} |f_m(z) - f_n(z)| &\leq |f_m(z) - f_m(z^*)| + |f_m(z^*) - f_n(z^*)| + |f_n(z^*) - f_n(z)| \\ &\leq \frac{\varepsilon}{3} + |f_m(z^*) - f_n(z^*)| + \frac{\varepsilon}{3}. \end{aligned}$$

On the other hand, by a. we know $(f_n(z^*))_{n \in \mathbf{N}}$ converges, hence is a Cauchy sequence. In other words, there exists $N_\varepsilon \in \mathbf{N}$ such that whenever $m \geq n \geq N_\varepsilon$, we have

$$|f_m(z^*) - f_n(z^*)| \leq \frac{\varepsilon}{3}.$$

For such choices of m and n , the previous two estimates yield

$$|f_m(z) - f_n(z)| \leq \varepsilon.$$

Hence $(f_n(z))_{n \in \mathbf{N}}$ is a Cauchy sequence, as claimed.

c. Given any $\varepsilon > 0$ and any $z \in K$, we choose $\delta_{\varepsilon,z}$ satisfying (0.1). Then the family of discs $\{B_{\delta_{\varepsilon,z}}(z) : z \in K\}$ is an open cover of the compact set K . By the (topological) definition of compactness there exists a finite subfamily $\{B_{\delta_{\varepsilon,z_i}}(z_i) : i = 1, \dots, N\}$ that still covers K . In other words, for every $z \in K$ there exists $i = 1, \dots, N$ such that $|z - z_i| < \delta_{\varepsilon,z_i}$.

Since the number of centers is finite, there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$, every $i = 1, \dots, N$ satisfies

$$|f(z_i) - f_n(z_i)| < \frac{\varepsilon}{3}.$$

Lastly, observe (0.1) also holds for the limit function f as we can pass to the limit in the estimate. Consequently, for $n \geq n_\varepsilon$ (independently of the point z to follow) we deduce that for every $z \in K$, we have

$$|f(z) - f_n(z)| \leq |f(z) - f(z_i)| + |f(z_i) - f_n(z_i)| + |f_n(z_i) - f_n(z)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The right-hand side is independent of z ; hence, taking the supremum over z on the left-hand side, we obtain uniform convergence of $(f_n)_{n \in \mathbb{N}}$ to f .

Since uniform convergence preserves continuity — one can e.g. pass to the limit in (0.1) — we conclude the proof.